Symmetries of Excited Heavy Baryons In The Heavy Quark And Large N_c Limit

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We demonstrate in a model independent way that, in the combined heavy quark and large N_c limit, there exists a new contracted U(4) symmetry which connects orbitally excited heavy baryons to the ground states.

Due to our inability to solve nonperturbative QCD from first principles, most of our quantitative understanding of low energy hadron properties are based on symmetry considerations. The most notable of these schemes is chiral perturbation theory, which is based on the fact that the QCD Lagrangian is approximately chirally invariant. On the other hand, there are emergent symmetries which are *not* symmetries (not even approximate symmetries) of the QCD Lagrangian, but emerge as symmetries of an effective theory obtained by taking certain limits. Two famous examples of such emergent symmetries, namely the heavy quark symmetry [1] and the large N_c spin-flavor symmetry [2], have important phenomenological implications and are well discussed in the literature.

In this paper, we will discuss a new emergent symmetry of QCD, which emerges in the heavy baryon (baryon with a single heavy quark) sector in the combined heavy quark and large N_c limit. As we will see below, this contracted U(4) symmetry connects the ground state baryon to some of its orbitally excited states. As a result, static properties like the axial current couplings and the moments of the weak form factors of these orbitally excited states can be related to their counterparts of the ground state. While some of these results have been discussed before in the literature, this is the first time where they are presented as symmetry predictions. Moreover, unlike previous studies, the analysis here is essentially model independent, depending only on the heavy quark and large N_c limits. We will outline the steps through which the existence of this symmetry can be demonstrated, while the details of the construction will be reported in the longer paper [3].

It has been pointed out that, in the combined heavy quark and large N_c limit, a heavy baryon can be regarded as a bound state of a heavy meson and a light baryon (a baryon for which all valence quarks are light) [4]. (Similar models are studied in [5].) For concreteness, we will always adopt the prescription that the heavy quark limit is taken before the large N_c limit. Since both constituents are infinitely massive (the heavy meson in the heavy quark limit, the light baryon in the large N_c limit), a small attraction between them is sufficient to ensure the existence of a bound state. By the usual large N_c counting rule it can be shown that the binding potential is of order N_c^0. Moreover, as the kinetic energy term is suppressed by the large reduced mass of the bound state, the wave function does not spread and is instead localized at the bottom of the potential. As a result, it can be approximated by a simple harmonic potential: $V(x) = V_0 + \frac{1}{2}\kappa\vec{x}^2$. By the large N_c counting rules, both V_0 and κ are of order N_c^0, and it has been shown that in the model studied in Refs. [4], $V_0 < 0$, $\kappa > 0$, *i.e.*, the potential is attractive and can support bound states. When the bound state is the ground state of the simple harmonic oscillator, it is a Λ_Q, the lightest heavy baryon containing the heavy quark Q. On the other hand, excited states in the simple harmonic oscillator are orbitally excited heavy baryons. We emphasize that this description of a heavy baryon is a model, which is *not* directly related to QCD.

After describing the physical picture of this model, which we will refer to as the bound state picture, we make the crucial observation that the excitation energy $\omega = \sqrt{\kappa/\mu}$ is small, where μ is the reduced mass of the bound state. By first taking the heavy quark limit, $\mu = m_N$ (mass of the light baryon) scales like N_c. Since the spring constant κ is of order N_c^0, ω scales like $N_c^{-1/2}$ and vanishes in the large N_c limit. This implies that when $N_c \to \infty$, the whole tower of excited states become degenerate with the ground state — a classic signature of an emergent symmetry.

What is the symmetry group of this emergent symmetry then? It has to contain, as a subgroup, the symmetry group of a three-dimensional simple harmonic oscillator, namely U(3) generated by $T_{ij} = a_i^\dagger a_j$ $(i, j = 1, 2, 3)$ where a_j is the annihilation operator in the j-th direction. These T_{ij}'s satisfy the U(3) commutation relations.

$$[T_{ij}, T_{kl}] = \delta_{il}T_{kj} + \delta_{kj}T_{il}. \tag{1}$$

When $N_c \to \infty$ and the excited states become degenerate with the ground state, the annihilation and creation operators a_j and $a_i^\dagger$ $(i, j = 1, 2, 3)$ also become generators of the emergent symmetry. The additional commutation relations are

$$[a_j, T_{kl}] = \delta_{kj}a_l, \quad [a_i^\dagger, T_{kl}] = -\delta_{il}a_k^\dagger, \quad [a_j, a_i^\dagger] = \delta_{ij}\mathbf{1}, \tag{2}$$

where $\mathbf{1}$ is the identity operator. These sixteen generators $\{T_{ij}, a_l, a_k^\dagger, \mathbf{1}\}$ form the *spectrum generating algebra* of a

three-dimensional harmonic oscillator. It is related to the usual U(4) algebra, generated by T_{ij} $(i, j = 1, 2, 3, 4)$ satisfying commutation relations (1) by the following limiting procedure:

$$a_j = \lim_{R\to\infty} T_{4j}/R, \quad a_i^\dagger = \lim_{R\to\infty} T_{i4}/R, \quad \mathbf{1} = \lim_{R\to\infty} T_{44}/R^2. \tag{3}$$

Such a limiting procedure is called a group contraction, and hence the group generated by $\{T_{ij}, a_l, a_k^\dagger, \mathbf{1}\}$ is called a *contracted* U(4) group.

So we have shown that the contracted U(4) is a symmetry of the bound state picture. But is it also a symmetry of QCD itself? We claim the answer is affirmative, and it is the aim of this paper to demonstrate the existence of this U(4) symmetry in QCD itself. We will first construct operators p_j and x_j $(j = 1, 2, 3)$, which correspond to the momentum and central position of the "brown muck" (light degrees of freedom) of the heavy baryon. From them operators a_j and $a_j^\dagger$ are constructed. By considering the double and triple commutator of the QCD Hamiltonian $\mathcal{H}$ with these operators, one can show that $\mathcal{H}$ is at most a bilinear in a_j and $a_j^\dagger$ as $N_c \to \infty$. Then it is straightforward to show that in the large N_c limit,

$$\mathcal{H} = H + H', \qquad H = \omega \sum_{j=1}^{3} a_j^\dagger a_j, \tag{4}$$

where ω is a parameter of order $N_c^{-1/2}$ and H' is an operator which commutes with all the a_j and $a_j^\dagger$. This Hamiltonian clearly has a contracted U(4) emergent symmetry in the large N_c limit as $\omega \to 0$.

Due to the conservation of baryon number and heavy quark number (in the heavy quark limit), it is well-defined to restrict our attention to the *heavy baryon Hilbert space*, *i.e.*, the subspace with both heavy quark number and baryon number equal to unity. In this subspace, we will define operators which correspond to the momenta and positions of the heavy quark and brown muck of the heavy baryon. So let $\mathcal{H}$ be the QCD Hamiltonian in this heavy baryon Hilbert space, P_j $(j = 1, 2, 3)$ be the momentum operators, and X_j be the operators conjugate to P_j. On the other hand, in the heavy quark limit, both the heavy quark mass m_Q and the heavy quark momentum operators P_{Q_j} $(j = 1, 2, 3)$ are well-defined up to order N_c^0 ambiguities, and X_{Q_j} are the operators conjugate to P_{Q_j}. These operators satisfy the following operator identities:

$$[\mathcal{H} X_j, \mathcal{H}] = iP_j, \qquad [m_Q X_{Q_j}, \mathcal{H}] = iP_{Q_j}, \tag{5}$$

where the first identity follows from Poincare invariance, and the second from heavy quark symmetry where Λ_{QCD}/m_Q corrections are dropped [1]. Note that in the heavy quark limit, both X_j and X_{Q_j} commute with $\mathcal{H}$ and hence are constants of motion. Lastly, we define the brown muck momentum operators $p_j = P_j - P_{Q_j}$, and x_j to be their conjugate operators.

The Hamiltonian H can be decomposed into three pieces: $\mathcal{H} = m_Q + m_N + \tilde{H}$, where by the large N_c scaling rules $m_N \sim N_c$ and $\tilde{H} \sim N_c^0$. Moreover, since $\mathcal{H} X_j = m_Q X_{Q_j} + (m_N + \tilde{H}) x_j$, one can subtract the operator identities (5) to obtain

$$[m_N x_j, \mathcal{H}] = [\mathcal{H} X_j - m_Q X_{Q_j}, \mathcal{H}] = i(P_j - P_{Q_j}) = ip_j, \tag{6}$$

with the term proportional to $\tilde{H}$ dropped as it is order N_c^{-1} suppressed relative to the leading order. So one has

$$[x_k, [x_j, \mathcal{H}]] = -\delta_{jk}/m_N \sim \mathcal{O}(N_c^{-1})(1 + \mathcal{O}(N_c^{-1})). \tag{7}$$

On the other hand, the double commutator $[p_k, [p_j, \mathcal{H}]] = [p_k, [p_j, \tilde{H}]]$ measures the second order energy change when the heavy quark is spatially moved with respect to the brown muck. Since $\tilde{H}$ is of order N_c^0, the double commutator is generically also of the same order. (For later use, we will mention in passing that, by the same logic, all multiple commutators like $[p_i, [p_j, \ldots [p_k, \mathcal{H}] \ldots]]$ are also of order N_c^0.) We will define

$$\hat{\kappa} = -[p_k, [p_j, \mathcal{H}]] \sim \mathcal{O}(N_c^0)(1 + \mathcal{O}(N_c^{-1})), \qquad \kappa = \langle G|\hat{\kappa}|G\rangle \tag{8}$$

where $|G\rangle$ is the ground state of QCD Hamiltonian $\mathcal{H}$.

Now let us take the heavy quark limit, and without loss of generality, set the constants of motion $X_j = X_{Q_j} = 0$, so that the heavy baryon is sitting at the origin, and x_j becomes the position of the center of the brown muck relative to the heavy quark, which is the center of mass of the heavy baryon. Then note that both Eqs. (7) and (8) will still

hold if the QCD Hamiltonian $\mathcal{H}$ in the double commutators are replaced by H, the Hamiltonian of a simple harmonic oscillator.

$$H = \sum_{j=1}^{3}\left[\frac{(p_j)^2}{2m_N} + \frac{\kappa(x_j)^2}{2} - \frac{\omega}{2}\right] = \omega\sum_{j=1}^{3} a_j^\dagger a_j, \qquad a_j = \sqrt{\frac{m_N\omega}{2}}x_j + i\sqrt{\frac{1}{2m_N\omega}}p_j, \tag{9}$$

where $a_j^\dagger$ is the hermitian conjugate of a_j and $\omega = \sqrt{\kappa/m_N} \sim \mathcal{O}(N_c^{-1/2})$. The contracted U(4) symmetry mentioned above is precisely the spectrum generating algebra of H and becomes an emergent symmetry as $\omega \to 0$ in the large N_c limit. On the other hand, to demonstrate that this contracted U(4) is a symmetry of QCD, one needs to show that the generators of the contracted U(4) commute with the QCD Hamiltonian $\mathcal{H}$, or equivalently, show that $\mathcal{H} = H + H'$, where H' commutes with a_j and $a_j^\dagger$ in the large N_c limit, *i.e.*, $[a_j, H'] = [a_j^\dagger, H'] = 0$.

Our strategy is to study all possible double and triple commutators of $\mathcal{H}$ with a_j and $a_j^\dagger$. It is straightforward to show that the vanishing of the triple commutators $[a_i,[a_j,[a_k,\mathcal{H}]]]$ and $[a_i^\dagger,[a_j,[a_k,\mathcal{H}]]]$ implies that $[a_k,\mathcal{H}] = Ca_k + Da_k^\dagger$, where both C and D commute with both a_k and $a_k^\dagger$. (A possible constant term is ruled out by parity.) But then $\mathcal{H}$ is at most a bilinear in a_k and $a_k^\dagger$:

$$\mathcal{H} = \sum_{k=1}^{3} Ca_k^\dagger a_k + D(a_k a_k + a_k^\dagger a_k^\dagger) + H', \tag{10}$$

where H' commutes with all the a_k and $a_k^\dagger$. Moreover, the forms of the operators C and D can be fixed by the relations $C\delta_{jk} = [a_j^\dagger,[a_k,\mathcal{H}]]$ and $D\delta_{jk} = [a_j,[a_k,\mathcal{H}]]$. Hence if $C = \omega$ and $D = 0$, $\mathcal{H} = H + H'$ with both terms invariant under the contracted U(4) symmetry, completing the argument that this is a symmetry of QCD.

The triple commutators $[a_i,[a_j,[a_k,\mathcal{H}]]]$ and $[a_i^\dagger,[a_j,[a_k,\mathcal{H}]]]$ are both linear combinations (with coefficients $\mathcal{O}(N_c^0)$) of these four terms:

$$\begin{aligned} T^{(0)} &= (m_N\omega)^{-3/2}[p_i,[p_j,[p_k,\mathcal{H}]]], \qquad & T^{(1)} &= (m_N\omega)^{-1/2}[p_i,[p_j,[x_k,\mathcal{H}]]], \\ T^{(2)} &= (m_N\omega)^{1/2}[p_i,[x_j,[x_k,\mathcal{H}]]], \qquad & T^{(3)} &= (m_N\omega)^{3/2}[x_i,[x_j,[x_k,\mathcal{H}]]]. \end{aligned} \tag{11}$$

As mentioned before, the triple p commutator is at most of order N_c^0, and hence $T^{(0)} \sim \mathcal{O}(N_c^{-3/4})$. All of the other three triple commutators are also small as $[x_k,\mathcal{H}] = ip_k/m_N + \mathcal{O}(N_c^{-2})$, and the first term does not contribute to the triple commutators. So $T^{(1)} \sim N_c^{-9/4}$, $T^{(2)} \sim N_c^{-7/4}$ and $T^{(3)} \sim N_c^{-5/4}$. All four terms are smaller than $\omega \sim N_c^{-1/2}$ in the large N_c limit, and hence so are the triple commutators $[a_i,[a_j,[a_k,\mathcal{H}]]]$ and $[a_i^\dagger,[a_j,[a_k,\mathcal{H}]]]$. By the strategy outlined above, the vanishings of these triple commutators in the large N_c limit imply that $\mathcal{H}$ is of the form in Eq. (10).

Lastly, we want to show that $C = \omega$ and $D = 0$, which is equivalent to showing that $[x_j,[x_j,\mathcal{H}]] = -1/m_N$ and $[p_j,[p_j,\mathcal{H}]] = -\kappa$. (The index j is not summed over.) While the former is true from Eq. (7), the latter is not: from Eq. (8) the double p commutator is $\hat{\kappa}$, which is in general not identical with its ground state expectation value κ. However, it is true for states in the *ground state band*, which is the subspace spanned by states of the form $(a_x^\dagger)^{n_x}(a_y^\dagger)^{n_y}(a_z^\dagger)^{n_z}|G\rangle$. So we conclude that, in the ground state band, $\mathcal{H}$ does have the form stated in Eq. (4), and hence is invariant under the contracted U(4) group in the large N_c limit. Note that this symmetry, like the familiar large N_c spin-flavor symmetry [2], only applies to a particular subspace of the theory — in this case the ground state band.

So we have demonstrated that this contracted U(4) symmetry is not only a symmetry of the bound state picture, but indeed a symmetry of QCD. Near the combined heavy quark and large N_c limit, there exists a band of low-lying heavy baryons, labeled by n_x, n_y and n_z, the number of excitation quanta in the x, y and z directions, and the excitation energies are $(n_x + n_y + n_z)\omega$. As $N_c \to \infty$, $\omega \to 0$ and the whole band become degenerate.

Such a symmetry has interesting phenomenological implications. For example, consider light quark form factors of the form $J_\ell = \bar{q}\Gamma q$, where Γ is an arbitrary combination of the gamma matrices, with momentum transfer of order $N_c\Lambda_{QCD}$. (In general the $\bar{q}$ and q may have different flavors.) Since J_ℓ is a single quark operator while the U(4) generators are collective coordinates involving N_c quarks, it follows that the commutators like $[a_j, J_\ell]$ are of order $\mathcal{O}(N_c^{-1})$ and vanish as $N_c \to \infty$. As a result, the light quark form factor is diagonal in the large N_c limit.

$$\langle\Lambda_Q^{(n_x',n_y',n_z')}(p')|J_\ell|\Lambda_Q^{(n_x,n_y,n_z)}(p)\rangle = \delta_{n_x n_x'}\delta_{n_y n_y'}\delta_{n_z n_z'}f(q^2), \qquad q^2 = (p-p')^2 \sim N_c^0\Lambda_{QCD}, \tag{12}$$

where $F(q^2)$ is an order N_c form factor, which is a function of q^2, the square of the momentum transfer. Consequently, $g_{\pi\Lambda_Q\Lambda_Q} \sim F(q^2)/f_\pi \sim N_c^{1/2}$, as the pion decay constant $f_\pi \sim N_c^{1/2}$. On the other hand, $g_{\pi\Lambda_Q\Lambda_Q^*}$ is suppressed in the

large N_c limit, a result which has been discussed in [6] in the context of the bound state picture, but here presented as an implication of the contracted U(4) group theory.

Like the light quark currents, heavy quark currents $J_h = \bar{Q}'\Gamma Q$ is also invariant under the contracted U(4) symmetry. Naively, one may conclude that the Isgur-Wise form factors, which are the matrix elements of such heavy quark currents, also only connect initial and final states with the same (n_x, n_y, n_z). Such conclusions are erroneous, however, as the final state is boosted relative to the initial state, and hence

$$\eta(w) \equiv \langle \Lambda_{Q'}^{(n'_x,n'_y,n'_z)}(v')|J_h|\Lambda_Q^{(n_x,n_y,n_z)}(v)\rangle = \langle \Lambda_{Q'}^{(n'_x,n'_y,n'_z)}(v)|B^\dagger_{v-v'}J_h|\Lambda_Q^{(n_x,n_y,n_z)}(v)\rangle, \tag{13}$$

where w is the scalar product of the four-velocities v and v', which is related to three-velocities v_j and v'_j by $w = 1 + |v_j - v'_j|^2/2$ in the non-relativistic limit. The boost operator $B_{v-v'} = \exp\left(i\mathcal{H}X_j(v-v')_j\right)$ boosts the heavy baryon from velocity v to v'. In the large N_c limit, $\mathcal{H}X_j = m_{Q'}X_{Q_j} + m_N x_j$, where the first term commutes with a_j and $a_j^\dagger$ but the second term does not. As a result, $B_{v-v'}$ does not commute with a_j and $a_j^\dagger$, and the Isgur-Wise form factors between states with different (n_x, n_y, n_z) do not vanish:

$$\eta(w) = \langle n'_x, n'_y, n_z|\exp\left(im_N x_j(v-v')_j\right)|n_x, n_y, n_z\rangle, \tag{14}$$

where $|n_x, n_y, n_z\rangle$ is the simple harmonic eigenstates, and $x_j = (a_j + a_j^\dagger)/\sqrt{2m_N\omega}$. This is simply a group theoretical expression which only depends on two parameters: m_N and ω, which can be fixed by measuring the excitation energy of the first excited state to be around 330 MeV. All Isgur-Wise form factors (or more exactly, all their derivatives at the point of zero recoil $w = 1$) between different initial and final heavy baryon states can be expressed as calculable functions of m_N and ω. For example, at the point of zero recoil ($v = v'$ and $w = 1$), the boost operator reduces to an identity operator and the Isgur-Wise form factors are non-zero if and only if $(n_x, n_y, n_z) = (n'_x, n'_y, n'_z)$; *i.e.*, the ground state Λ_Q can only decay into a ground state $\Lambda_{Q'}$, a well-known prediction of heavy quark symmetry [1]. When the velocity transfer is non-zero but small, $w = 1 + \epsilon$, the ground state Λ_Q can decay into excited $\Lambda_{Q'}$. Since x_j is linear in $a_j^\dagger$, however, at order ϵ the only non-vanishing excited state Isgur-Wise form factor is that to the first excited state, and it saturates both the Bjorken and Voloshin sum rules [6]. This analysis of the Isgur-Wise form factors recasts the studies of Refs. [4,6] in a model independent way.

In the above, we have ignored the spins and flavors of the heavy baryons. The inclusion of these quantum numbers does not change the above analysis. In particular, both the spin-flavor symmetry for the brown muck and the flavor symmetry for the heavy quark, being generated by one-quark operators, commute with our contracted U(4) in the large N_c limit. These extra excitation modes live in the H' term in Eq. (4). For example, with two light flavors, $H' = \sigma I^2$, with $\sigma \sim N_c^{-1}$ for states with $I + s_\ell = 0$, where I is isospin and s_ℓ is the spin of brown muck (excluding any possible orbital angular momentum) [7]. So the eigenstates of H' is a tower of states labeled by $I = 0, 1, \ldots$, where the states with $I = 0$ and 1 are the Λ_Q and $\Sigma_Q^{(*)}$, respectively. Since $\sigma \ll \omega$ in the large N_c limit, inclusion of such a H' splits each simple harmonic eigenstate of H into a whole tower of $I = s_\ell$ states.

In conclusion, we have demonstrated that in the combined heavy quark and large N_c limit, there exists a new emergent symmetry which connects orbitally excited states to the ground states. While such a symmetry has interesting phenomenological implications, its utility for quantitative predictions may be limited by potentially large corrections which are typically of order $N_c^{-1/2}$. For example, the effect of an anharmonic term ax^4 ($a \sim N_c^0$) leads to order $N_c^{-1/2}$ mixings between the simple harmonic oscillator states. However, even though the corrections may be large, the existence of such a symmetry provides a useful organization principle for low energy properties of heavy baryons, and may provide qualitative or semi-quantitative predictions.

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